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1 In a party with 1982 persons, among any group of four there is at least one person who knows each of the other three. What is the minimum number of people in the party who know everyone else?

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2 Let $X_r = x^r + y^r + z^r$ with x, y, z real. It is known that if $S_1 = 0$,

$$(*) \quad \frac{S_{m+n}}{m+n} = \frac{S_m}{m}\frac{S_n}{n}$$

for $(m,n) = (2,3), (3,2), (2,5)$, or $(5,2)$. Determine *all* other pairs of integers (m,n) if any, so that $(*)$ holds for all real numbers x, y, z such that $x + y + z = 0$.

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3 If a point A_1 is in the interior of an equilateral triangle ABC and point A_2 is in the interior of $\triangle A_1BC$, prove that

$$\mathrm{I.\,Q.}(A_1BC) > \mathrm{I.\,Q.}(A_2BC),$$

where the *isoperrimetric quotient* of a figure F is defined by

$$\mathrm{I.\,Q.}(F) = \frac{\mathrm{Area}(F)}{[\mathrm{Perimeter}(F)]^2}.$$

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4 Prove that there exists a positive integer k such that $k \cdot 2^n + 1$ is composite for every integer n.

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5 A, B, and C are three interior points of a sphere S such that AB and AC are perpendicular to the diameter of S through A, and so that two spheres can be constructed through A, B, and C which are both tangent to S. Prove that the sum of their radii is equal to the radius of S.
