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1 On a given circle, six points A, B, C, D, E, and F are chosen at random, independently and uniformly with respect to arc length. Determine the probability that the two triangles ABC and DEF are disjoint, i.e., have no common points.

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2 Prove that the roots of

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$$

cannot all be real if $2a^2 < 5b$.

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3 Each set of a finite family of subsets of a line is a union of two closed intervals. Moreover, any three of the sets of the family have a point in common. Prove that there is a point which is common to at least half the sets of the family.

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4 Six segments $S_1, S_2, S_3, S_4, S_5,$ and S_6 are given in a plane. These are congruent to the edges $AB, AC, AD, BC, BD,$ and CD, respectively, of a tetrahedron $ABCD$. Show how to construct a segment congruent to the altitude of the tetrahedron from vertex A with straight-edge and compasses.

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5 Consider an open interval of length $1/n$ on the real number line, where n is a positive integer. Prove that the number of irreducible fractions p/q, with $1 \leq q \leq n$, contained in the given interval is at most $(n+1)/2$.

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