ITAMO 2017

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1 Let a and b be positive real numbers. Consider a regular hexagon of side a, and build externally on its sides six rectangles of sides a and b. The new twelve vertices lie on a circle. Now repeat the same construction, but this time exchanging the roles of a and b; namely; we start with a regular hexagon of side b and we build externally on its sides six rectangles of sides a and b. The new twelve vertices lie on another circle.
Show that the two circles have the same radius.

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2 Let $n \geq 2$ be an integer. Consider the solutions of the system

$$\begin{cases} n = a + b - c \\ n = a^2 + b^2 - c^2 \end{cases}$$

where a, b, c are integers. Show that there is at least one solution and that the solutions are finitely many.

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3 Madam Mim has a deck of 52 cards, stacked in a pile with their backs facing up. Mim separates the small pile consisting of the seven cards on the top of the deck, turns it upside down, and places it at the bottom of the deck. All cards are again in one pile, but not all of them face down; the seven cards at the bottom do, in fact, face up. Mim repeats this move until all cards have their backs facing up again. In total, how many moves did Mim make $?$

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4 Let $ABCD$ be a thetraedron with the following propriety: the four lines connecting a vertex and the incenter of opposite face are concurrent. Prove $AB \cdot CD = AC \cdot BD = AD \cdot BC$.

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5 Let $x_1, x_2, x_3 \ldots$ a succession of positive integers such that for every couple of positive integers (m, n) we have $x_{mn} \neq x_{m(n+1)}$. Prove that there exists a positive integer i such that $x_i \geq 2017$.

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6 Prove that there are infinitely many positive integers m such that the number of odd distinct prime factor of $m(m+3)$ is a multiple of 3.
