Benelux 2017

www.artofproblemsolving.com/community/c448392

by socrates

1 Find all functions $f : \mathbb{Q}_{>0} \to \mathbb{Z}_{>0}$ such that

$$f(xy) \cdot \gcd\left(f(x)f(y), f(\frac{1}{x})f(\frac{1}{y})\right) = xyf(\frac{1}{x})f(\frac{1}{y}),$$

for all $x, y \in \mathbb{Q}_{>0,}$ where $\gcd(a, b)$ denotes the greatest common divisor of a and b.

---

2 Let $n \geq 2$ be an integer. Alice and Bob play a game concerning a country made of n islands. Exactly two of those n islands have a factory. Initially there is no bridge in the country. Alice and Bob take turns in the following way. In each turn, the player must build a bridge between two different islands I_1 and I_2 such that: • I_1 and I_2 are not already connected by a bridge. • at least one of the two islands I_1 and I_2 is connected by a series of bridges to an island with a factory (or has a factory itself). (Indeed, access to a factory is needed for the construction.) As soon as a player builds a bridge that makes it possible to go from one factory to the other, this player loses the game. (Indeed, it triggers an industrial battle between both factories.) If Alice starts, then determine (for each $n \geq 2$) who has a winning strategy.
(*Note:* It is allowed to construct a bridge passing above another bridge.)

---

3 In the convex quadrilateral $ABCD$ we have $\angle B = \angle C$ and $\angle D = 90^\circ$. Suppose that $|AB| = 2|CD|$. Prove that the angle bisector of $\angle ACB$ is perpendicular to CD.

---

4 A *Benelux n-square* (with $n \geq 2$) is an $n \times n$ grid consisting of n^2 cells, each of them containing a positive integer, satisfying the following conditions: • the n^2 positive integers are pairwise distinct. • if for each row and each column we compute the greatest common divisor of the n numbers in that row/column, then we obtain $2n$ different outcomes.

(a) Prove that, in each Benelux n-square (with $n \geq 2$), there exists a cell containing a number which is at least $2n^2$.
(b) Call a Benelux n-square *minimal* if all n^2 numbers in the cells are at most $2n^2$. Determine all $n \geq 2$ for which there exists a minimal Benelux n-square.
