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1   (a) Do there exist 14 consecutive positive integers each of which is divisible by one or more primes p from the interval $2 \leq p \leq 11$?

(b) Do there exist 21 consecutive positive integers each of which is divisible by one or more primes p from the interval $2 \leq p \leq 13$?

2   During a certain lecture, each of five mathematicians fell asleep exactly twice. For each pair of mathematicians, there was some moment when both were asleep simultaneously. Prove that, at some moment, three of them were sleeping simultaneously.

3   What is the smallest integer n, greater than one, for which the root-mean-square of the first n positive integers is an integer?

Note. The root-mean-square of n numbers $a_1, a_2, \cdots, a_n$ is defined to be

$$\left[\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} \right]^{1/2}$$

4   Two distinct circles K_1 and K_2 are drawn in the plane. They intersect at points A and B, where AB is the diameter of K_1. A point P on K_2 and inside K_1 is also given.

Using only a "T-square" (i.e. an instrument which can produce a straight line joining two points and the perpendicular to a line through a point on or off the line), find a construction for two points C and D on K_1 such that CD is perpendicular to AB and $\angle CPD$ is a right angle.

5   By a partition π of an integer $n \geq 1$, we mean here a representation of n as a sum of one or more positive integers where the summands must be put in nondecreasing order. (E.g., if $n = 4$, then the partitions π are $1 + 1 + 1 + 1, 1 + 1 + 2, 1 + 3, 2 + 2$, and 4).

For any partition π, define $A(\pi)$ to be the number of 1's which appear in π, and define $B(\pi)$ to be the number of distinct integers which appear in π. (E.g., if $n = 13$ and π is the partition $1 + 1 + 2 + 2 + 2 + 5$, then $A(\pi) = 2$ and $B(\pi) = 3$).

Prove that, for any fixed n, the sum of $A(\pi)$ over all partitions of π of n is equal to the sum of $B(\pi)$ over all partitions of π of n.

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