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by Binomial-theorem, rrusczyk

1 Determine all solutions in non-zero integers a and b of the equation

$$(a^2+b)(a+b^2)=(a-b)^3.$$

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2 AD, BE, and CF are the bisectors of the interior angles of triangle ABC, with D, E, and F lying on the perimeter. If angle EDF is 90 degrees, determine all possible values of angle BAC.

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3 Construct a set S of polynomials inductively by the rules:

(i) $x\in S$;
(ii) if $f(x)\in S$, then $xf(x)\in S$ and $x+(1-x)f(x)\in S$.

Prove that there are no two distinct polynomials in S whose graphs intersect within the region $\{0<x<1\}$.

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4 Three circles C_i are given in the plane: C_1 has diameter AB of length 1; C_2 is concentric and has diameter k ($1<k<3$); C_3 has center A and diameter $2k$. We regard k as fixed. Now consider all straight line segments XY which have one endpoint X on C_2, one endpoint Y on C_3, and contain the point B. For what ratio XB/BY will the segment XY have minimal length?

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5 Given a sequence $(x_1,x_2,\ldots,x_n)$ of 0's and 1's, let A be the number of triples (x_i,x_j,x_k) with $i<j<k$ such that (x_i,x_j,x_k) equals $(0,1,0)$ or $(1,0,1)$. For $1\le i\le n$, let d_i denote the number of j for which either $j<i$ and $x_j=x_i$ or else $j>i$ and $x_j\ne x_i$.

(a) Prove that

$$A=\binom{n}{3}-\sum_{i=1}^{n}\binom{d_i}{2}.$$

(Of course, $\binom{a}{b}=\frac{a!}{b!(a-b)!}$.) [5 points]

(b) Given an odd number n, what is the maximum possible value of A? [15 points]

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