

USAMO 1989

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1 For each positive integer n , let

$$\begin{aligned}S_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \\T_n &= S_1 + S_2 + S_3 + \cdots + S_n, \\U_n &= \frac{T_1}{2} + \frac{T_2}{3} + \frac{T_3}{4} + \cdots + \frac{T_n}{n+1}.\end{aligned}$$

Find, with proof, integers $0 < a, b, c, d < 1000000$ such that $T_{1988} = aS_{1989} - b$ and $U_{1988} = cS_{1989} - d$.

2 The 20 members of a local tennis club have scheduled exactly 14 two-person games among themselves, with each member playing in at least one game. Prove that within this schedule there must be a set of 6 games with 12 distinct players.

3 Let $P(z) = z^n + c_1z^{n-1} + c_2z^{n-2} + \cdots + c_n$ be a polynomial in the complex variable z , with real coefficients c_k . Suppose that $|P(i)| < 1$. Prove that there exist real numbers a and b such that $P(a + bi) = 0$ and $(a^2 + b^2 + 1)^2 < 4b^2 + 1$.

4 Let ABC be an acute-angled triangle whose side lengths satisfy the inequalities $AB < AC < BC$. If point I is the center of the inscribed circle of triangle ABC and point O is the center of the circumscribed circle, prove that line IO intersects segments AB and BC .

5 Let u and v be real numbers such that

$$(u + u^2 + u^3 + \cdots + u^8) + 10u^9 = (v + v^2 + v^3 + \cdots + v^{10}) + 10v^{11} = 8.$$

Determine, with proof, which of the two numbers, u or v , is larger.

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