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1	Find, as a function of $n,$ the sum of the digits of $$9 \times 99 \times 9999 \times \cdots \times \left(10^{2^n} - 1\right),$$ where each factor has twice as many digits as the previous one.
2	Prove $$\frac{1}{\cos 0^\circ \cos 1^\circ} + \frac{1}{\cos 1^\circ \cos 2^\circ} + \cdots + \frac{1}{\cos 88^\circ \cos 89^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}.$$
3	For a nonempty set S of integers, let $\sigma(S)$ be the sum of the elements of S. Suppose that $A = \{a_1, a_2, \ldots, a_{11}\}$ is a set of positive integers with $a_1 < a_2 < \cdots < a_{11}$ and that, for each positive integer $n \leq 1500,$ there is a subset S of A for which $\sigma(S) = n$. What is the smallest possible value of a_{10}?
4	Chords AA', BB', CC' of a sphere meet at an interior point P but are not contained in a plane. The sphere through A, B, C, P is tangent to the sphere through A', B', C', P. Prove that $AA' = BB' = CC'$.
5	Let $P(z)$ be a polynomial with complex coefficients which is of degree 1992 and has distinct zeros. Prove that there exist complex numbers $a_1, a_2, \ldots, a_{1992}$ such that $P(z)$ divides the polynomial $$\left(\cdots\left((z-a_1)^2 - a_2\right)^2 \cdots - a_{1991}\right)^2 - a_{1992}.$$
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