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Day 1 April 28th

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1 Suppose that the set $\{1,2,\cdots,1998\}$ has been partitioned into disjoint pairs $\{a_i,b_i\}$ ($1\le i\le 999$) so that for all i, $|a_i-b_i|$ equals 1 or 6. Prove that the sum

$$|a_1-b_1|+|a_2-b_2|+\cdots+|a_{999}-b_{999}|$$

ends in the digit 9.

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2 Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent AB to $\mathcal{C}_2$ ($B\in\mathcal{C}_2$). Let C be the second point of intersection of AB and $\mathcal{C}_1$, and let D be the midpoint of AB. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

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3 Let $a_0,a_1,\cdots,a_n$ be numbers from the interval $(0,\pi/2)$ such that

$$\tan(a_0-\frac{\pi}{4})+\tan(a_1-\frac{\pi}{4})+\cdots+\tan(a_n-\frac{\pi}{4})\ge n-1.$$

Prove that

$$\tan a_0\tan a_1\cdots\tan a_n\ge n^{n+1}.$$

Day 2 April 28th

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4 A computer screen shows a 98×98 chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white, white becomes black). Find, with proof, the minimum number of mouse clicks needed to make the chessboard all one color.

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5 Prove that for each $n\ge 2$, there is a set S of n integers such that $(a-b)^2$ divides ab for every distinct $a,b\in S$.

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6 Let $n\ge 5$ be an integer. Find the largest integer k (as a function of n) such that there exists a convex n-gon $A_1A_2\dots A_n$ for which exactly k of the quadrilaterals $A_iA_{i+1}A_{i+2}A_{i+3}$ have an inscribed circle. (Here $A_{n+j}=A_j$.)
