**USAMO 2003**

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**Day 1** April 29th

**1** Prove that for every positive integer $n$ there exists an $n$-digit number divisible by $5^n$ all of whose digits are odd.

**2** A convex polygon $\mathcal{P}$ in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygon $\mathcal{P}$ are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.

**3** Let $n \neq 0$. For every sequence of integers

$$A = a_0, a_1, a_2, \dots, a_n$$

satisfying $0 \le a_i \le i$, for $i = 0, \dots, n$, define another sequence

$$t(A) = t(a_0), t(a_1), t(a_2), \dots, t(a_n)$$

by setting $t(a_i)$ to be the number of terms in the sequence $A$ that precede the term $a_i$ and are different from $a_i$. Show that, starting from any sequence $A$ as above, fewer than $n$ applications of the transformation $t$ lead to a sequence $B$ such that $t(B) = B$.

**Day 2** April 30th

**4** Let $ABC$ be a triangle. A circle passing through $A$ and $B$ intersects segments $AC$ and $BC$ at $D$ and $E$, respectively. Lines $AB$ and $DE$ intersect at $F$, while lines $BD$ and $CF$ intersect at $M$. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

**5** Let $a$, $b$, $c$ be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \le 8.$$

**6** At the vertices of a regular hexagon are written six nonnegative integers whose sum is $2003^{2003}$. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

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