**USAMO 2008**

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**Day 1** April 29th

**1** Prove that for each positive integer $n$, there are pairwise relatively prime integers $k_0, k_1, \dots, k_n$, all strictly greater than $1$, such that $k_0 k_1 \dots k_n - 1$ is the product of two consecutive integers.

**2** Let $ABC$ be an acute, scalene triangle, and let $M$, $N$, and $P$ be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray $AM$ in points $D$ and $E$ respectively, and let lines $BD$ and $CE$ intersect in point $F$, inside of triangle $ABC$. Prove that points $A$, $N$, $F$, and $P$ all lie on one circle.

**3** Let $n$ be a positive integer. Denote by $S_n$ the set of points $(x, y)$ with integer coordinates such that

$$|x| + \left| y + \frac{1}{2} \right| < n.$$

A path is a sequence of distinct points $(x_1, y_1), (x_2, y_2), \dots, (x_\ell, y_\ell)$ in $S_n$ such that, for $i = 2, \dots, \ell$, the distance between $(x_i, y_i)$ and $(x_{i-1}, y_{i-1})$ is $1$ (in other words, the points $(x_i, y_i)$ and $(x_{i-1}, y_{i-1})$ are neighbors in the lattice of points with integer coordinates). Prove that the points in $S_n$ cannot be partitioned into fewer than $n$ paths (a partition of $S_n$ into $m$ paths is a set $\mathcal{P}$ of $m$ nonempty paths such that each point in $S_n$ appears in exactly one of the $m$ paths in $\mathcal{P}$).

**Day 2** April 30th

**4** Let $\mathcal{P}$ be a convex polygon with $n$ sides, $n \geq 3$. Any set of $n - 3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a *triangulation* of $\mathcal{P}$ into $n - 2$ triangles. If $\mathcal{P}$ is regular and there is a triangulation of $\mathcal{P}$ consisting of only isosceles triangles, find all the possible values of $n$.

**5** Three nonnegative real numbers $r_1$, $r_2$, $r_3$ are written on a blackboard. These numbers have the property that there exist integers $a_1$, $a_2$, $a_3$, not all zero, satisfying $a_1 r_1 + a_2 r_2 + a_3 r_3 = 0$. We are permitted to perform the following operation: find two numbers $x$, $y$ on the blackboard with $x \leq y$, then erase $y$ and write $y - x$ in its place. Prove that after a finite number of such operations, we can end up with at least one $0$ on the blackboard.

**6** At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e., is of the form $2^k$ for some positive integer $k$).

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