**USAMO 2014**

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by patrickhompe, msinghal, djmathman, tenniskidperson3, rrusczyk

**Day 1** April 29th

**1** Let $a$, $b$, $c$, $d$ be real numbers such that $b-d \geq 5$ and all zeros $x_1, x_2, x_3,$ and $x_4$ of the polynomial $P(x)=x^4+ax^3+bx^2+cx+d$ are real. Find the smallest value the product $(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1)$ can take.

**2** Let $\mathbb{Z}$ be the set of integers. Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that

$$xf(2f(y)-x)+y^2f(2x-f(y))=\frac{f(x)^2}{x}+f(yf(y))$$

for all $x, y \in \mathbb{Z}$ with $x \neq 0$.

**3** Prove that there exists an infinite set of points

$$\dots,\ P_{-3},\ P_{-2},\ P_{-1},\ P_0,\ P_1,\ P_2,\ P_3,\ \dots$$

in the plane with the following property: For any three distinct integers $a, b,$ and $c$, points $P_a$, $P_b$, and $P_c$ are collinear if and only if $a+b+c=2014$.

**Day 2** April 30th

**4** Let $k$ be a positive integer. Two players $A$ and $B$ play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with $A$ moving first. In his move, $A$ may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, $B$ may choose any counter on the board and remove it. If at any time there are $k$ consecutive grid cells in a line all of which contain a counter, $A$ wins. Find the minimum value of $k$ for which $A$ cannot win in a finite number of moves, or prove that no such minimum value exists.

**5** Let $ABC$ be a triangle with orthocenter $H$ and let $P$ be the second intersection of the circumcircle of triangle $AHC$ with the internal bisector of the angle $\angle BAC$. Let $X$ be the circumcenter of triangle $APB$ and $Y$ the orthocenter of triangle $APC$. Prove that the length of segment $XY$ is equal to the circumradius of triangle $ABC$.

**6** Prove that there is a constant $c>0$ with the following property: If $a, b, n$ are positive integers such that $\gcd(a+i, b+j)>1$ for all $i, j \in \{0, 1, \dots n\}$, then

$$\min\{a, b\} > c^n \cdot n^{\frac{n}{2}}.$$

 