Pan African 2002

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Day 1

1 Find all functions $f : N_0 \to N_0$, (where N_0 is the set of all non-negative integers) such that $f(f(n)) = f(n) + 1$ for all $n \in N_0$ and the minimum of the set $\{f(0), f(1), f(2) \cdots\}$ is 1.

2 $\triangle AOB$ is a right triangle with $\angle AOB = 90^o$. C and D are moving on AO and BO respectively such that $AC = BD$. Show that there is a fixed point P through which the perpendicular bisector of CD always passes.

3 Prove for every integer $n > 0$, there exists an integer $k > 0$ such that $2^n k$ can be written in decimal notation using only digits 1 and 2.

Day 2

4 Seven students in a class compare their marks in 12 subjects studied and observe that no two of the students have identical marks in all 12 subjects. Prove that we can choose 6 subjects such that any two of the students have different marks in at least one of these subjects.

5 Let $\triangle ABC$ be an acute angled triangle. The circle with diameter AB intersects the sides AC and BC at points E and F respectively. The tangents drawn to the circle through E and F intersect at P.
Show that P lies on the altitude through the vertex C.

6 If $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and $a_1 + a_2 + \cdots + a_n = 1$, then prove:

$$a_1^2 + 3a_2^2 + 5a_3^2 + \cdots + (2n-1)a_n^2 \leq 1$$