Pan African 2006

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by djb86

Day 1

1 Let AB and CD be two perpendicular diameters of a circle with centre O. Consider a point M on the diameter AB, different from A and B. The line CM cuts the circle again at N. The tangent at N to the circle and the perpendicular at M to AM intersect at P. Show that $OP = CM$.

2 Let a, b, c be three non-zero integers. It is known that the sums $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ and $\frac{b}{a} + \frac{c}{b} + \frac{a}{c}$ are integers. Find these sums.

3 For a real number x let $\lfloor x \rfloor$ be the greatest integer less than or equal to x and let $\{x\} = x - \lfloor x \rfloor$. If a, b, c are distinct real numbers, prove that

$$\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-a)(b-c)} + \frac{c^3}{(c-a)(c-b)}$$

is an integer if and only if $\{a\} + \{b\} + \{c\}$ is an integer.

Day 2

4 For every positive integer k let $a(k)$ be the largest integer such that $2^{a(k)}$ divides k. For every positive integer n determine $a(1) + a(2) + \cdots + a(2^n)$.

5 In how many ways can the integers from 1 to 2006 be divided into three non-empty disjoint sets so that none of these sets contains a pair of consecutive integers?

6 Let ABC be a right angled triangle at A. Denote D the foot of the altitude through A and O_1, O_2 the incentres of triangles ADB and ADC. The circle with centre A and radius AD cuts AB in K and AC in L. Show that O_1, O_2, K and L are on a line.