Pan African 2007

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Day 1

1 Find all natural numbers N consisting of exactly 1112 digits (in decimal notation) such that:
(a) The sum of the digits of N is divisible by 2000;
(b) The sum of the digits of $N+1$ is divisible by 2000;
(c) 1 is a digit of N.

2 Let A, B and C be three fixed points, not on the same line. Consider all triangles $AB'C'$ where B' moves on a given straight line (not containing A), and C' is determined such that $\angle B' = \angle B$ and $\angle C' = \angle C$. Find the locus of C'.

3 In a country, towns are connected by roads. Each town is directly connected to exactly three other towns. Show that there exists a town from which you can make a round-trip, without using the same road more than once, and for which the number of roads used is not divisible by 3. (Not all towns need to be visited.)

Day 2

1 Solve the following system of equations for real x, y and z:

$$\begin{aligned} x &= \sqrt{2y+3} \\ y &= \sqrt{2z+3} \\ z &= \sqrt{2x+3}. \end{aligned}$$

2 For which positive integers n is $231^n - 222^n - 8^n - 1$ divisible by 2007?

3 An equilateral triangle of side length 2 is divided into four pieces by two perpendicular lines that intersect in the centroid of the triangle. What is the maximum possible area of a piece?