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by WakeUp

Day 1

1	Determine all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying $f(x+y)\le f(x)+f(y)\le x+y$ for all $x,y\in\mathbb{R}$.
2	Let C_1 be a circle with centre O, and let AB be a chord of the circle that is not a diameter. M is the midpoint of AB. Consider a point T on the circle C_2 with diameter OM. The tangent to C_2 at the point T intersects C_1 at two points. Let P be one of these points. Show that $PA^2+PB^2=4PT^2$.
3	Let a,b,c be three positive integers such that $a<b<c$. Consider the the sets A,B,C and X, defined as follows: $A=\{1,2,\ldots,a\}$, $B=\{a+1,a+2,\ldots,b\}$, $C=\{b+1,b+2,\ldots,c\}$ and $X=A\cup B\cup C$. Determine, in terms of a,b and c, the number of ways of placing the elements of X in three boxes such that there are x,y and z elements in the first, second and third box respectively, knowing that: i) $x\le y\le z$; ii) elements of B cannot be put in the first box; iii) elements of C cannot be put in the third box.

Day 2

1	Let x and y be two positive reals. Prove that $xy\le\frac{x^{n+2}+y^{n+2}}{x^n+y^n}$ for all non-negative integers n.
2	A set of positive integers X is called *connected* if $\|X\|\ge 2$ and there exist two distinct elements m and n of X such that m is a divisor of n. Determine the number of connected subsets of the set $\{1,2,\ldots,10\}$.
3	Prove that for all positive integers n, there exists a positive integer m which is a multiple of n and the sum of the digits of m is equal to n.