Pan African 2010

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by WakeUp

Day 1

1 a) Show that it is possible to pair off the numbers $1, 2, 3, \ldots, 10$ so that the sums of each of the five pairs are five different prime numbers.
b) Is it possible to pair off the numbers $1, 2, 3, \ldots, 20$ so that the sums of each of the ten pairs are ten different prime numbers?

2 How many ways are there to line up 19 girls (all of different heights) in a row so that no girl has a shorter girl both in front of and behind her?

3 In an acute-angled triangle ABC, CF is an altitude, with F on AB, and BM is a median, with M on CA. Given that $BM = CF$ and $\angle MBC = \angle FCA$, prove that triangle ABC is equilateral.

Day 2

1 Seven distinct points are marked on a circle of circumference c. Three of the points form an equilateral triangle and the other four form a square. Prove that at least one of the seven arcs into which the seven points divide the circle has length less than or equal $\frac{c}{24}$.

2 A sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ of positive integers is constructed as follows:
-if the last digit of a_n is less than or equal to 5 then this digit is deleted and a_{n+1} is the number consisting of the remaining digits. (If a_{n+1} contains no digits the process stops.)
-otherwise $a_{n+1} = 9a_n$.
Can one choose a_0 so that an infinite sequence is obtained?

3 Does there exist a function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x + f(y)) = f(x) - y$ for all integers x and y?