Pan African 2013

www.artofproblemsolving.com/community/c4525

by djb86

Day 1 June 28th

1 A positive integer n is such that $n(n+2013)$ is a perfect square.
a) Show that n cannot be prime.
b) Find a value of n such that $n(n+2013)$ is a perfect square.

2 Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x)f(y)+f(x+y)=xy$ for all real numbers x and y.

3 Let $ABCDEF$ be a convex hexagon with $\angle A=\angle D$ and $\angle B=\angle E$. Let K and L be the midpoints of the sides AB and DE respectively. Prove that the sum of the areas of triangles FAK, KCB and CFL is equal to half of the area of the hexagon if and only if

$$\frac{BC}{CD}=\frac{EF}{FA}.$$

Day 2

1 Let $ABCD$ be a convex quadrilateral with AB parallel to CD. Let P and Q be the midpoints of AC and BD, respectively. Prove that if $\angle ABP=\angle CBD$, then $\angle BCQ=\angle ACD$.

2 The cells of an $n\times n$ board with $n\geq 5$ are coloured black or white so that no three adjacent squares in a row, column or diagonal are the same colour. Show that for any 3×3 square within the board, two of its corner squares are coloured black and two are coloured white.

3 Let x, y, and z be real numbers such that $x<y<z<6$. Solve the system of inequalities:

$$\begin{cases} \dfrac{1}{y-x}+\dfrac{1}{z-y}\leq 2 \\ \dfrac{1}{6-z}+2\leq x \end{cases}$$