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Day 1

1 Find the function $f(x)$ such that

$$f(x)^2 f\left(\frac{1-x}{x+1}\right) = 64x$$

for $x \neq 0, x \neq 1, x \neq -1$.

2 In a triangle ABC, M and N are the respective midpoints of the sides AC and AB, and P is the point of intersection of BM and CN. Prove that, if it is possible to inscribe a circle in the quadrilateral $AMPN$, then the triangle ABC is isosceles.

3 Prove that if m, n, r are positive integers, and:

$$1 + m + n\sqrt{3} = (2 + \sqrt{3})^{2r-1}$$

then m is a perfect square.

Day 2

1 The sequence (p_n) is defined as follows: $p_1 = 2$ and for all n greater than or equal to 2, p_n is the largest prime divisor of the expression $p_1 p_2 p_3 \ldots p_{n-1} + 1$.
Prove that every p_n is different from 5.

2 Let r, s, t be the roots of the equation $x(x-2)(3x-7) = 2$. Show that r, s, t are real and positive and determine $\arctan r + \arctan s + \arctan t$.

3 Let $ABCD$ be a convex quadrilateral and let P and Q be the points on the sides AD and BC respectively such that $\frac{AP}{PD} = \frac{BQ}{QC} = \frac{AB}{CD}$.
Prove that the line PQ forms equal angles with the lines AB and CD.