IberoAmerican 1990

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Day 1 September 24th

1 Let f be a function defined for the non-negative integers, such that:

a) $f(n)=0$ if $n=2^j-1$ for some $j\geq 0$.
b) $f(n+1)=f(n)-1$ otherwise.

i) Show that for every $n\geq 0$ there exists $k\geq 0$ such that $f(n)+n=2^k-1$.
ii) Find $f(2^{1990})$.

2 Let ABC be a triangle. I is the incenter, and the incircle is tangent to BC, CA, AB at D, E, F, respectively. P is the second point of intersection of AD and the incircle. If M is the midpoint of EF, show that P, I, M, D are concyclic.

3 Let b, c be integer numbers, and define $f(x)=(x+b)^2-c$.

i) If p is a prime number such that c is divisible by p but not by p^2, show that for every integer n, $f(n)$ is not divisible by p^2.

ii) Let $q\neq 2$ be a prime divisor of c. If q divides $f(n)$ for some integer n, show that for every integer r there exists an integer n' such that $f(n')$ is divisible by qr.

Day 2 September 25th

4 Let Γ_1 be a circle. AB is a diameter, ℓ is the tangent at B, and M is a point on Γ_1 other than A. Γ_2 is a circle tangent to ℓ, and also to Γ_1 at M.

a) Determine the point of tangency P of ℓ and Γ_2 and find the locus of the center of Γ_2 as M varies.

b) Show that there exists a circle that is always orthogonal to Γ_2, regardless of the position of M.

5 A and B are two opposite vertices of an $n\times n$ board. Within each small square of the board, the diagonal parallel to AB is drawn, so that the board is divided in $2n^2$ equal triangles. A coin moves from A to B along the grid, and for every segment of the grid that it visits, a seed is put in each triangle that contains the segment as a side. The path followed by the coin is such that no segment is visited more than once, and after the coins arrives at B, there are exactly two seeds in each of the $2n^2$ triangles of the board. Determine all the values of n for which such scenario is possible.

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6 Let $f(x)$ be a cubic polynomial with rational coefficients. If the graph of $f(x)$ is tangent to the x axis, prove that the roots of $f(x)$ are all rational.
