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Day 1

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1 A number n is said to be *nice* if it exists an integer $r > 0$ such that the expression of n in base r has all
its digits equal. For example, 62 and 15 are *nice* because 62 is 222 in base 5, and 15 is 33 in base 4. Show that 1993 is not *nice*, but 1994 is.

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2 Let $ABCD$ a cuadrilateral inscribed in a circumference. Suppose that there is a semicircle with its center on AB, that
is tangent to the other three sides of the cuadrilateral.

(i) Show that $AB = AD + BC$.

(ii) Calculate, in term of $x = AB$ and $y = CD$, the maximal area that can be reached for such quadrilateral.

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3 In each square of an $n \times n$ grid there is a lamp. If the lamp is touched it changes its state every lamp in the same row and every lamp in the same column (the one that are on are turned off and viceversa). At the begin, all the lamps are off. Show that lways is possible, with an appropriated sequence of touches, that all the the lamps on the board end on and find, in function of n the minimal number of touches that are necessary to turn on every lamp.

Day 2

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1 Let $A,\ B$ and C be given points on a circumference K such that the triangle $\triangle ABC$ is acute. Let P be a point in the interior of K. $X,\ Y$ and Z be the other intersection of AP, BP and CP with the circumference. Determine the position of P such that $\triangle XYZ$ is equilateral.

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2 Let n and r two positive integers. It is wanted to make r subsets $A_1,\ A_2, \dots, A_r$ from the set $\{0, 1, \cdots, n-1\}$ such that all those subsets contain exactly k elements and such that, for all integer x with $0 \leq x \leq n-1$ there exist $x_1 \in A_1,\ x_2 \in A_2 \dots, x_r \in A_r$ (an element of each set) with $x = x_1 + x_2 + \cdots + x_r$.

Find the minimum value of k in terms of n and r.

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3 Show that every natural number $n \leq 2^{1\ 000\ 000}$ can be obtained first with 1 doing less than $1\ 100\ 000$ sums; more precisely, there is a finite sequence of natural numbers $x_0,\ x_1, \dots,\ x_k$ with $k \leq$

$1\ 100\ 000,\ x_0 = 1,\ x_k = n$ such that for all $i = 1,\ 2, \ldots,\ k$ there exist $r,\ s$ with $0 \leq r \leq s < i$ such that $x_i = x_r + x_s$.