IberoAmerican 1995

www.artofproblemsolving.com/community/c4535

by carlosbr, Pascual2005, parmenides51

Day 1

1 Find all the possible values of the sum of the digits of all the perfect squares.

[Commented by djimenez]
Comment: I would rewrite it as follows:
Let $f:\mathbb{N}\rightarrow\mathbb{N}$ such that $f(n)$ is the sum of all the digits of the number n^2. Find the image of f (where, by image it is understood the set of all x such that exists an n with $f(n)=x$).

2 Let n be a positive integer greater than 1. Determine all the collections of real numbers $x_1,\ x_2,\dots,\ x_n\geq 1$ and $x_{n+1}\leq 0$ such that the next two conditions hold:

(i) $x_1^{\frac12}+x_2^{\frac32}+\cdots+x_n^{n-\frac12}= nx_{n+1}^{\frac12}$

(ii) $\frac{x_1+x_2+\cdots+x_n}{n}=x_{n+1}$

3 Let r and s two orthogonal lines that does not lay on the same plane. Let AB be their common perpendicular, where $A\in r$ and $B\in s$(*).Consider the sphere of diameter AB. The points $M\in r$ and $N\in s$ varies with the condition that MN is tangent to the sphere on the point T. Find the locus of T.

Note: The plane that contains B and r is perpendicular to s.

Day 2

1 In a $m\times n$ grid are there are token. Every token *dominates* every square on its same row ($\leftrightarrow$), its same column ($\updownarrow$), and diagonal ($\nwarrow\!\!\searrow$)(Note that the token does not *dominate* the diagonal ($\swarrow\!\!\nearrow$), determine the lowest number of tokens that must be on the board to *dominate* all the squares on the board.

2 The incircle of a triangle ABC touches the sides BC, CA, AB at the points D, E, F respectively. Let the line AD intersect this incircle of triangle ABC at a point X (apart from D). Assume that this point X is the midpoint of the segment AD, this means, $AX=XD$. Let the line BX meet the incircle of triangle ABC at a point Y (apart from X), and let the line CX meet the incircle of triangle ABC at a point Z (apart from X). Show that $EY=FZ$.

3 A function $f:N\rightarrow N$ is circular if for every $p\in N$ there exists $n\in N,\ n\leq p$ such that $f^n(p)=p$ (f composed with itself n times) The function f has repulsion degree $k>0$ if for every $p\in N$

$f^i(p) \neq p$ for every $i = 1, 2, \dots, \lfloor kp \rfloor$. Determine the maximum repulsion degree can have a circular function.

Note: Here $\lfloor x \rfloor$ is the integer part of x.