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by carlosbr, parmenides51

Day 1

1 Let n be a natural number. A cube of edge n may be divided in 1996 cubes whose edges length are also natural numbers. Find the minimum possible value for n.

2 Let $\triangle ABC$ be a triangle, D the midpoint of BC, and M be the midpoint of AD. The line BM intersects the side AC on the point N. Show that AB is tangent to the circuncircle to the triangle $\triangle NBC$ if and only if the following equality is true:

$$\frac{BM}{MN}=\frac{(BC)^2}{(BN)^2}.$$

3 We have a grid of k^2-k+1 rows and k^2-k+1 columns, where $k=p+1$ and p is prime. For each prime p, give a method to put the numbers 0 and 1, one number for each square in the grid, such that on each row there are exactly k 0's, on each column there are exactly k 0's, and there is no rectangle with sides parallel to the sides of the grid with 0s on each four vertices.

Day 2

1 Given a natural number $n\geq 2$, consider all the fractions of the form $\frac{1}{ab}$, where a and b are natural numbers, relative primes and such that: $a<b\leq n$, $a+b>n$.
Show that for each n, the sum of all this fractions are $\frac{1}{2}$.

2 Three tokens A, B, C are, each one in a vertex of an equilateral triangle of side n. Its divided on equilateral triangles of side 1, such as it is shown in the figure for the case $n=3$

Initially, all the lines of the figure are painted blue. The tokens are moving along the lines painting them of red, following the next two rules:

(1) First A moves, after that B moves, and then C, by turns. On each turn, the token moves over exactly one line of one of the little triangles, form one side to the other.
(2) Non token moves over a line that is already painted red, but it can rest on one endpoint of a side that is already red, even if there is another token there waiting its turn.

Show that for every positive integer n it is possible to paint red all the sides of the little triangles.

3 There are n different points $A_1, ..., A_n$ in the plane and at each point A_i is assigned a real number λ_i, distinct from zero, in such way that $(\overline{A_iA_j})^2 = \lambda_i + \lambda_j$ for all i, j with $i \neq j$. Show that:
(1) $n \leq 4$
(2) If $n = 4$, then $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} = 0$