IberoAmerican 1997

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Day 1

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1 Let $r \geq 1$ be a real number that holds with the property that for each pair of positive integer numbers m and n, with n a multiple of m, it is true that $\lfloor nr \rfloor$ is multiple of $\lfloor mr \rfloor$. Show that r has to be an integer number.

Note: If x is a real number, $\lfloor x \rfloor$ is the greatest integer lower than or equal to x.

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2 In a triangle ABC, it is drawn a circumference with center in the incenter I and that meet twice each of the sides of the triangle: the segment BC on D and P (where D is nearer two B); the segment CA on E and Q (where E is nearer to C); and the segment AB on F and R (where F is nearer to A).

Let S be the point of intersection of the diagonals of the quadrilateral $EQFR$. Let T be the point of intersection of the diagonals of the quadrilateral $FRDP$. Let U be the point of intersection of the diagonals of the quadrilateral $DPEQ$.

Show that the circumcircle to the triangle $\triangle FRT$, $\triangle DPU$ and $\triangle EQS$ have a unique point in common.

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3 Let $n \geq 2$ be an integer number and D_n the set of all the points (x, y) in the plane such that its coordinates are integer numbers with: $-n \leq x \leq n$ and $-n \leq y \leq n$.

(a) There are three possible colors in which the points of D_n are painted with (each point has a unique color). Show that with
any distribution of the colors, there are always two points of D_n with the same color such that the line that contains them does not go through any other point of D_n.

(b) Find a way to paint the points of D_n with 4 colors such that if a line contains exactly two points of D_n, then, this points have different colors.

Day 2

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1 Let n be a positive integer. Consider the sum $x_1y_1 + x_2y_2 + \cdots + x_ny_n$, where that values of the variables $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ are either 0 or 1.

Let $I(n)$ be the number of 2n-tuples $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$ such that the sum of the number is odd, and let $P(n)$ be the number of 2n-tuples $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$ such that the sum is an even number. Show that:

$$\frac{P(n)}{I(n)} = \frac{2^n + 1}{2^n - 1}$$

2 In an acute triangle $\triangle ABC$, let AE and BF be highs of it, and H its orthocenter. The symmetric line of AE with respect to the angle bisector of $\sphericalangle A$ and the symmetric line of BF with respect to the angle bisector of $\sphericalangle B$ intersect each other on the point O. The lines AE and AO intersect again the circuncircle to $\triangle ABC$ on the points M and N respectively.

Let P be the intersection of BC with HN; R the intersection of BC with OM; and S the intersection of HR with OP. Show that $AHSO$ is a paralelogram.

3 Let $P = \{P_1, P_2, ..., P_{1997}\}$ be a set of 1997 points in the interior of a circle of radius 1, where P_1 is the center of the circle. For each $k = 1. \dots, 1997$, let x_k be the distance of P_k to the point of P closer to P_k, but different from it. Show that $(x_1)^2 + (x_2)^2 + ... + (x_{1997})^2 \leq 9$.