IberoAmerican 1999

www.artofproblemsolving.com/community/c4539

by carlosbr

Day 1

1 Find all the positive integers less than 1000 such that the cube of the sum of its digits is equal to the square of such integer.

2 Given two circle M and N, we say that M bisects N if they intersect in two points and the common chord is a diameter of N. Consider two fixed non-concentric circles C_1 and C_2.
a) Show that there exists infinitely many circles B such that B bisects both C_1 and C_2.
b) Find the locus of the centres of such circles B.

3 Let $P_1, P_2, \dots, P_n$ be n distinct points over a line in the plane ($n \geq 2$). Consider all the circumferences with diameters P_iP_j ($1 \leq i, j \leq n$) and they are painted with k given colors. Lets call this configuration a (n, k)-cloud.

For each positive integer k, find all the positive integers n such that every possible (n, k)-cloud has two mutually exterior tangent circumferences of the same color.

Day 2

1 Let B be an integer greater than 10 such that everyone of its digits belongs to the set $\{1, 3, 7, 9\}$. Show that B has a **prime divisor** greater than or equal to 11.

2 An acute triangle $\triangle ABC$ is inscribed in a circle with centre O. The altitudes of the triangle are AD, BE and CF. The line EF cut the circumference on P and Q.
a) Show that OA is perpendicular to PQ.
b) If M is the midpoint of BC, show that $AP^2 = 2AD \cdot OM$.

3 Let A and B points in the plane and C a point in the perpendiclar bisector of AB. It is constructed a sequence of points $C_1, C_2, \dots, C_n, \dots$ in the following way: $C_1 = C$ and for $n \geq 1$, if C_n does not belongs to AB, then C_{n+1} is the circumcentre of the triangle $\triangle ABC_n$.

Find all the points C such that the sequence $C_1, C_2, \dots$ is defined for all n and turns eventually periodic.

Note: A sequence $C_1, C_2, \dots$ is called eventually periodic if there exist positive integers k and p such that $C_{n+p} = c_n$ for all $n \geq k$.