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Day 1

1 The integer numbers from 1 to 2002 are written in a blackboard in increasing order $1, 2, \ldots, 2001, 2002$. After that, somebody erases the numbers in the $(3k+1)-th$ places i.e. $(1, 4, 7, \ldots)$. After that, the same person erases the numbers in the $(3k+1)-th$ positions of the new list (in this case, $2, 5, 9, \ldots$). This process is repeated until one number remains. What is this number?

2 Given any set of 9 points in the plane such that there is no 3 of them collinear, show that for each point P of the set, the number of triangles with its vertices on the other 8 points and that contain P on its interior is even.

3 Let P be a point in the interior of the equilateral triangle $\triangle ABC$ such that $\sphericalangle APC = 120^\circ$. Let M be the intersection of CP with AB, and N the intersection of AP and BC. Find the locus of the circumcentre of the triangle MBN as P varies.

Day 2

1 In a triangle $\triangle ABC$ with all its sides of different length, D is on the side AC, such that BD is the angle bisector of $\sphericalangle ABC$. Let E and F, respectively, be the feet of the perpendicular drawn from A and C to the line BD and let M be the point on BC such that DM is perpendicular to BC. Show that $\sphericalangle EMD = \sphericalangle DMF$.

2 The sequence of real numbers $a_1, a_2, \ldots$ is defined as follows: $a_1 = 56$ and $a_{n+1} = a_n - \frac{1}{a_n}$ for $n \geq 1$. Show that there is an integer $1 \leq k \leq 2002$ such that $a_k < 0$.

3 A policeman is trying to catch a robber on a board of 2001×2001 squares. They play alternately, and the player whose trun it is moves to a space in one of the following directions: $\downarrow, \rightarrow, \nwarrow$.

If the policeman is on the square in the bottom-right corner, he can go directly to the square in the upper-left corner (the robber can not do this). Initially the policeman is in the central square, and the robber is in the upper-left adjacent square. Show that:

$a)$ The robber may move at least 10000 times before the being captured. $b)$ The policeman has an strategy such that he will eventually catch the robber.

Note: The policeman can catch the robber if he reaches the square where the robber is, but not if the robber enters the square occupied by the policeman.