IberoAmerican 2003

www.artofproblemsolving.com/community/c4543

by carlosbr

Day 1

1 (a)There are two sequences of numbers, with 2003 consecutive integers each, and a table of 2 rows and 2003 columns

			$\cdots\cdots$		
			$\cdots\cdots$		

Is it always possible to arrange the numbers in the first sequence in the first row and the second sequence in the second row, such that the sequence obtained of the 2003 column-wise sums form a new sequence of 2003 consecutive integers? (b) What if 2003 is replaced with 2004?

2 Let C and D be two points on the semicricle with diameter AB such that B and C are on distinct sides of the line AD. Denote by M, N and P the midpoints of AC, BD and CD respectively. Let O_A and O_B the circumcentres of the triangles ACP and BDP. Show that the lines O_AO_B and MN are parallel.

3 Pablo copied from the blackboard the problem:
Consider all the sequences of 2004 real numbers $(x_0, x_1, x_2, \dots, x_{2003})$ such that: $x_0 = 1, 0 \leq x_1 \leq 2x_0, 0 \leq x_2 \leq 2x_1 \dots, 0 \leq x_{2003} \leq 2x_{2002}$. From all these sequences, determine the sequence which minimizes $S = \cdots$

As Pablo was copying the expression, it was erased from the board. The only thing that he could remember was that S was of the form $S = \pm x_1 \pm x_2 \pm \cdots \pm x_{2002} + x_{2003}$. Show that, even when Pablo does not have the complete statement, he can determine the solution of the problem.

Day 2

1 Let $M = \{1, 2, \dots, 49\}$ be the set of the first 49 positive integers. Determine the maximum integer k such that the set M has a subset of k elements such that there is no 6 consecutive integers in such subset. For this value of k, find the number of subsets of M with k elements with the given property.

2 In a square $ABCD$, let P and Q be points on the sides BC and CD respectively, different from its endpoints, such that $BP = CQ$. Consider points X and Y such that $X \neq Y$, in the segments AP and AQ respectively. Show that, for every X and Y chosen, there exists a triangle whose sides have lengths BX, XY and DY.

3 The sequences $(a_n), (b_n)$ are defined by $a_0 = 1, b_0 = 4$ and for $n \geq 0$

$$a_{n+1} = a_n^{2001} + b_n, \;\; b_{n+1} = b_n^{2001} + a_n$$

Show that 2003 is not divisor of any of the terms in these two sequences.