IberoAmerican 2004

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Day 1

1 It is given a 1001*1001 board divided in 1*1 squares. We want to amrk m squares in such a way that:
1: if 2 squares are adjacent then one of them is marked.
2: if 6 squares lie consecutively in a row or column then two adjacent squares from them are marked.

Find the minimun number of squares we most mark.

2 In the plane are given a circle with center O and radius r and a point A outside the circle. For any point M on the circle, let N be the diametrically opposite point. Find the locus of the circumcenter of triangle AMN when M describes the circle.

3 Let n and k be positive integers such as either n is odd or both n and k are even. Prove that exists integers a and b such as $GCD(a,n) = GCD(b,n) = 1$ and $k = a + b$

Day 2

1 Determine all pairs (a, b) of positive integers, each integer having two decimal digits, such that $100a + b$ and $201a + b$ are both perfect squares.

2 Given a scalene triangle ABC. Let A', B', C' be the points where the internal bisectors of the angles CAB, ABC, BCA meet the sides BC, CA, AB, respectively. Let the line BC meet the perpendicular bisector of AA' at A''. Let the line CA meet the perpendicular bisector of BB' at B'. Let the line AB meet the perpendicular bisector of CC' at C''. Prove that A'', B'' and C'' are collinear.

3 Given a set $\mathcal{H}$ of points in the plane, P is called an "intersection point of $\mathcal{H}$" if distinct points A, B, C, D exist in $\mathcal{H}$ such that lines AB and CD are distinct and intersect in P.
Given a finite set $\mathcal{A}_0$ of points in the plane, a sequence of sets is defined as follows: for any $j \geq 0$, $\mathcal{A}_{j+1}$ is the union of $\mathcal{A}_j$ and the intersection points of $\mathcal{A}_j$.
Prove that, if the union of all the sets in the sequence is finite, then $\mathcal{A}_i = \mathcal{A}_1$ for any $i \geq 1$.