IberoAmerican 2005

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Day 1 September 27th

1 Determine all triples of real numbers (a, b, c) such that

$$\begin{aligned} xyz &= 8 \\ x^2y + y^2z + z^2x &= 73 \\ x(y-z)^2 + y(z-x)^2 + z(x-y)^2 &= 98. \end{aligned}$$

2 A flea jumps in a straight numbered line. It jumps first from point 0 to point 1. Afterwards, if its last jump was from A to B, then the next jump is from B to one of the points $B + (B - A) - 1$, $B + (B - A)$, $B + (B - A) + 1$.

Prove that if the flea arrived twice at the point n, n positive integer, then it performed at least $\lceil 2\sqrt{n} \rceil$ jumps.

3 Let $p > 3$ be a prime. Prove that if

$$\sum_{i=1}^{p-1} \frac{1}{i^p} = \frac{n}{m},$$

with $gcd\ (n, m) = 1$, then p^3 divides n.

Day 2 September 28th

4 Denote by $a \bmod b$ the remainder of the euclidean division of a by b. Determine all pairs of positive integers (a, p) such that p is prime and

$$a \bmod p + a \bmod 2p + a \bmod 3p + a \bmod 4p = a + p.$$

5 Let O be the circumcenter of acutangle triangle ABC and let A_1 be some point in the smallest arc BC of the circumcircle of ABC. Let A_2 and A_3 points on sides AB and AC, respectively, such that $\angle BA_1A_2 = \angle OAC$ and $\angle CA_1A_3 = \angle OAB$.

Prove that the line A_2A_3 passes through the orthocenter of ABC.

6 Let n be a fixed positive integer. The points A_1, A_2, $\ldots$, A_{2n} are on a straight line. Color each point blue or red according to the following procedure: draw n pairwise disjoint circumferences, each with diameter A_iA_j for some $i \neq j$ and such that every point A_k belongs to exactly one circumference. Points in the same circumference must be of the same color.

Determine the number of ways of coloring these $2n$ points when we vary the n circumferences and the distribution of the colors.