Serbia Team Selection Test 2017

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– Day 1

1 Let ABC be a triangle and D the midpoint of the side BC. Define points E and F on AC and B, respectively, such that $DE = DF$ and $\angle EDF = \angle BAC$. Prove that

$$DE \geq \frac{AB + AC}{4}.$$

2 Initally a pair (x, y) is written on the board, such that exactly one of it's coordinates is odd. On such a pair we perform an operation to get pair $(\frac{x}{2}, y + \frac{x}{2})$ if $2|x$ and $(x + \frac{y}{2}, \frac{y}{2})$ if $2|y$. Prove that for every odd $n > 1$ there is a even positive integer $b < n$ such that starting from the pair (n, b) we will get the pair (b, n) after finitely many operations.

3 A function $f : \mathbb{N} \to \mathbb{N}$ is called nice if $f^a(b) = f(a + b - 1)$, where $f^a(b)$ denotes a times applied function f.
Let g be a nice function, and an integer A exists such that $g(A + 2018) = g(A) + 1$.
a) Prove that $g(n + 2017^{2017}) = g(n)$ for all $n \geq A + 2$.
b) If $g(A + 1) \neq g(A + 1 + 2017^{2017})$ find $g(n)$ for $n < A$.

– Day 2

4 We have an $n \times n$ square divided into unit squares. Each side of unit square is called unit segment. Some isoceles right triangles of hypotenuse 2 are put on the square so all their vertices are also vertices of unit squares. For which n it is possible that every unit segment belongs to exactly one triangle(unit segment belongs to a triangle even if it's on the border of the triangle)?

5 Let $n \geq 2$ be a positive integer and $\{x_i\}_{i=0}^n$ a sequence such that not all of its elements are zero and there is a positive constant C_n for which:
(i) $x_1 + \cdots + x_n = 0$, and
(ii) for each i either $x_i \leq x_{i+1}$ or $x_i \leq x_{i+1} + C_n x_{i+2}$ (all indexes are assumed modulo n).
Prove that
a) $C_n \geq 2$, and
b) $C_n = 2$ if and only $2 \mid n$.

6 Let k be a positive integer and let n be the smallest number with exactly k divisors. Given n is a cube, is it possible that k is divisible by a prime factor of the form $3j + 2$?

