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Day 1 September 22nd

1 Given a positive integer $n \geq 2$, consider a set of n islands arranged in a circle. Between every two neigboring islands two bridges are built as shown in the figure.

Starting at the island X_1, in how many ways one can one can cross the $2n$ bridges so that no bridge is used more than once?

2 Define the succession a_n, $n > 0$ as $n + m$, where m is the largest integer such that $2^{2^m} \leq n2^n$. Find all numbers that are not in the succession.

3 Let C_1 and C_2 be two congruent circles centered at O_1 and O_2, which intersect at A and B. Take a point P on the arc AB of C_2 which is contained in C_1. AP meets C_1 at C, CB meets C_2 at D and the bisector of $\angle CAD$ intersects C_1 and C_2 at E and L, respectively. Let F be the symmetric point of D with respect to the midpoint of PE. Prove that there exists a point X satisfying $\angle XFL = \angle XDC = 30^\circ$ and $CX = O_1O_2$.

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Day 2 September 23rd

4 Given a triangle ABC of incenter I, let P be the intersection of the external bisector of angle A and the circumcircle of ABC, and J the second intersection of PI and the circumcircle of ABC. Show that the circumcircles of triangles JIB and JIC are respectively tangent to IC and IB.

5 Consider the sequence $\{a_n\}_{n\geq 1}$ defined as follows: $a_1 = 1$, $a_{2k} = 1 + a_k$ and $a_{2k+1} = \frac{1}{a_{2k}}$ for every $k \geq 1$. Prove that every positive rational number appears on the sequence $\{a_n\}$ exactly once.

6 Six thousand points are marked on a circle, and they are colored using 10 colors in such a way that within every group of 100 consecutive points all the colors are used. Determine the least positive integer k with the following property: In every coloring satisfying the condition above, it is possible to find a group of k consecutive points in which all the colors are used.