IberoAmerican 2011

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Day 1

1 The number 2 is written on the board. Ana and Bruno play alternately. Ana begins. Each one, in their turn, replaces the number written by the one obtained by applying exactly one of these operations: multiply the number by 2, multiply the number by 3 or add 1 to the number. The first player to get a number greater than or equal to 2011 wins. Find which of the two players has a winning strategy and describe it.

2 Find all positive integers n for which exist three nonzero integers x, y, z such that $x+y+z=0$ and:

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{n}$$

3 Let ABC be a triangle and X, Y, Z be the tangency points of its inscribed circle with the sides BC, CA, AB, respectively. Suppose that C_1, C_2, C_3 are circle with chords YZ, ZX, XY, respectively, such that C_1 and C_2 intersect on the line CZ and that C_1 and C_3 intersect on the line BY. Suppose that C_1 intersects the chords XY and ZX at J and M, respectively; that C_2 intersects the chords YZ and XY at L and I, respectively; and that C_3 intersects the chords YZ and ZX at K and N, respectively. Show that I, J, K, L, M, N lie on the same circle.

Day 2

1 Let ABC be an acute-angled triangle, with $AC \neq BC$ and let O be its circumcenter. Let P and Q be points such that $BOAP$ and $COPQ$ are parallelograms. Show that Q is the orthocenter of ABC.

2 Let $x_1, \ldots, x_n$ be positive real numbers. Show that there exist $a_1, \ldots, a_n \in \{-1, 1\}$ such that:

$$a_1x_1^2+a_2x_2^2+\ldots+a_nx_n^2 \geq (a_1x_1+a_2x_2+\ldots+a_nx_n)^2$$

3 Let k and n be positive integers, with $k \geq 2$. In a straight line there are kn stones of k colours, such that there are n stones of each colour. A *step* consists of exchanging the position of two adjacent stones. Find the smallest positive integer m such that it is always possible to achieve, with at most m steps, that the n stones are together, if:
a) n is even.
b) n is odd and $k=3$