IberoAmerican 2012

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Day 1

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1 Let $ABCD$ be a rectangle. Construct equilateral triangles BCX and DCY, in such a way that both of these triangles share some of their interior points with some interior points of the rectangle. Line AX intersects line CD on P, and line AY intersects line BC on Q. Prove that triangle APQ is equilateral.

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2 A positive integer is called *shiny* if it can be written as the sum of two not necessarily distinct integers a and b which have the same sum of their digits. For instance, 2012 is *shiny*, because $2012 = 2005 + 7$, and both 2005 and 7 have the same sum of their digits. Find all positive integers which are **not** *shiny* (the dark integers).

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3 Let n to be a positive integer. Given a set $\{a_1, a_2, \ldots, a_n\}$ of integers, where $a_i \in \{0, 1, 2, 3, \ldots, 2^n - 1\}$, $\forall i$, we associate to each of its subsets the sum of its elements; particularly, the empty subset has sum of its elements equal to 0. If all of these sums have different remainders when divided by 2^n, we say that $\{a_1, a_2, \ldots, a_n\}$ is [i]n-complete[/i].

For each n, find the number of [i]n-complete[/i] sets.

Day 2

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1 Let a, b, c, d be integers such that the number $a - b + c - d$ is odd and it divides the number $a^2 - b^2 + c^2 - d^2$. Show that, for every positive integer n, $a - b + c - d$ divides $a^n - b^n + c^n - d^n$.

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2 Let ABC be a triangle, P and Q the intersections of the parallel line to BC that passes through A with the external angle bisectors of angles B and C, respectively. The perpendicular to BP at P and the perpendicular to CQ at Q meet at R. Let I be the incenter of ABC. Show that $AI = AR$.

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3 Show that, for every positive integer n, there exist n consecutive positive integers such that none is divisible by the sum of its digits.

(Alternative Formulation: Call a number good if it's not divisible by the sum of its digits. Show that for every positive integer n there are n consecutive good numbers.)
