IberoAmerican 2014

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Day 1 September 23rd

1 For each positive integer n, let $s(n)$ be the sum of the digits of n. Find the smallest positive integer k such that

$$s(k) = s(2k) = s(3k) = \cdots = s(2013k) = s(2014k).$$

2 Find all polynomials $P(x)$ with real coefficients such that $P(2014) = 1$ and, for some integer c:

$xP(x-c) = (x-2014)P(x)$

3 2014 points are placed on a circumference. On each of the segments with end points on two of the 2014 points is written a non-negative real number. For any convex polygon with vertices on some of the 2014 points, the sum of the numbers written on their sides is less or equal than 1. Find the maximum possible value for the sum of all the written numbers.

Day 2 September 24th

1 N coins are placed on a table, $N-1$ are genuine and have the same weight, and one is fake, with a different weight. Using a two pan balance, the goal is to determine with certainty the fake coin, and whether it is lighter or heavier than a genuine coin. Whenever one can deduce that one or more coins are genuine, they will be inmediately discarded and may no longer be used in subsequent weighings. Determine all N for which the goal is achievable. (There are no limits regarding how many times one may use the balance).

Note: the only difference between genuine and fake coins is their weight; otherwise, they are identical.

2 Let ABC be an acute triangle and H its orthocenter. Let D be the intersection of the altitude from A to BC. Let M and N be the midpoints of BH and CH, respectively. Let the lines DM and DN intersect AB and AC at points X and Y respectively. If P is the intersection of XY with BH and Q the intersection of XY with CH, show that H, P, D, Q lie on a circumference.

3 Given a set X and a function $f : X \to X$, for each $x \in X$ we define $f^1(x) = f(x)$ and, for each $j \geq 1$, $f^{j+1}(x) = f(f^j(x))$. We say that $a \in X$ is a fixed point of f if $f(a) = a$. For each $x \in \mathbb{R}$, let $\pi(x)$ be the quantity of positive primes lesser or equal to x.

Given an positive integer n, we say that $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ is *catracha* if $f^{f(k)}(k) = k$, for every $k = 1, 2, \dots n$. Prove that:

(a) If f is catracha, f has at least $\pi(n) - \pi(\sqrt{n}) + 1$ fixed points.
(b) If $n \geq 36$, there exists a catracha function f with exactly $\pi(n) - \pi(\sqrt{n}) + 1$ fixed points.