CentroAmerican 2003

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1 Two players A and B take turns playing the following game: There is a pile of 2003 stones. In his first turn, A selects a divisor of 2003 and removes this number of stones from the pile. B then chooses a divisor of the number of remaining stones, and removes that number of stones from the new pile, and so on. The player who has to remove the last stone loses. Show that one of the two players has a winning strategy and describe the strategy.

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2 S is a circle with AB a diameter and t is the tangent line to S at B. Consider the two points C and D on t such that B is between C and D. Suppose E and F are the intersections of S with AC and AD and G and H are the intersections of S with CF and DE. Show that $AH = AG$.

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3 Let a and b be positive integers with $a > 1$ and $b > 2$. Prove that $a^b + 1 \geq b(a+1)$ and determine when there is inequality.

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4 S_1 and S_2 are two circles that intersect at two different points P and Q. Let ℓ_1 and ℓ_2 be two parallel lines such that ℓ_1 passes through the point P and intersects S_1, S_2 at A_1, A_2 respectively (both distinct from P), and ℓ_2 passes through the point Q and intersects S_1, S_2 at B_1, B_2 respectively (both distinct from Q).
Show that the triangles A_1QA_2 and B_1PB_2 have the same perimeter.

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5 A square board with 8cm sides is divided into 64 squares square with each side 1cm. Each box can be painted white or black. Find the total number of ways to colour the board so that each square of side 2cm formed by four squares with a common vertex contains two white and two black squares.

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6 Say a number is *tico* if the sum of it's digits is a multiple of 2003.

(i) Show that there exists a positive integer N such that the first 2003 multiples, $N, 2N, 3N, \dots 2003N$ are all tico.

(ii) Does there exist a positive integer N such that all it's multiples are tico?
