CentroAmerican 2004

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by WakeUp

Day 1

1 On a whiteboard, the numbers 1 to 9 are written. Players A and B take turns, and A is first. Each player in turn chooses one of the numbers on the whiteboard and removes it, along with all multiples (if any). The player who removes the last number loses.
Determine whether any of the players has a winning strategy, and explain why.

2 Define the sequence (a_n) as follows: $a_0=a_1=1$ and for $k\ge 2$, $a_k=a_{k-1}+a_{k-2}+1$.
Determine how many integers between 1 and 2004 inclusive can be expressed as a_m+a_n with m and n positive integers and $m\neq n$.

3 ABC is a triangle, and E and F are points on the segments BC and CA respectively, such that $\frac{CE}{CB}+\frac{CF}{CA}=1$ and $\angle CEF=\angle CAB$. Suppose that M is the midpoint of EF and G is the point of intersection between CM and AB. Prove that triangle FEG is similar to triangle ABC.

Day 2

1 In a 10×10 square board, half of the squares are coloured white and half black. One side common to two squares on the board side is called a *border* if the two squares have different colours. Determine the minimum and maximum possible number of borders that can be on the board.

2 Let $ABCD$ be a trapezium such that $AB||CD$ and $AB+CD=AD$. Let P be the point on AD such that $AP=AB$ and $PD=CD$.

$a)$ Prove that $\angle BPC=90^{\circ}$. $b)$ Q is the midpoint of BC and R is the point of intersection between the line AD and the circle passing through the points B,A and Q. Show that the points B,P,R and C are concyclic.

3 With pearls of different colours form necklaces, it is said that a necklace is *prime* if it cannot be decomposed into strings of pearls of the same length, and equal to each other.
Let n and q be positive integers. Prove that the number of prime necklaces with n beads, each of which has one of the q^n possible colours, is equal to n times the number of prime necklaces with n^2 pearls, each of which has one of q possible colours.

Note: two necklaces are considered equal if they have the same number of pearls and you can get the same colour on both necklaces, rotating one of them to match it to the other.