CentroAmerican 2006

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Day 1 August 1st

1 For $0 \leq d \leq 9$, we define the numbers

$$S_d = 1 + d + d^2 + \cdots + d^{2006}$$

Find the last digit of the number

$$S_0 + S_1 + \cdots + S_9.$$

2 Let Γ and Γ' be two congruent circles centered at O and O', respectively, and let A be one of their two points of intersection. B is a point on Γ, C is the second point of intersection of AB and Γ', and D is a point on Γ' such that $OBDO'$ is a parallelogram. Show that the length of CD does not depend on the position of B.

3 For every natural number n we define

$$f(n) = \left\lfloor n + \sqrt{n} + \frac{1}{2} \right\rfloor$$

Show that for every integer $k \geq 1$ the equation

$$f(f(n)) - f(n) = k$$

has exactly $2k - 1$ solutions.

Day 2 August 2nd

4 The product of several distinct positive integers is divisible by 2006^2. Determine the minimum value the sum of such numbers can take.

5 The *Olimpia* country is formed by n islands. The most populated one is called *Panacenter*, and every island has a different number of inhabitants. We want to build bridges between these islands, which we'll be able to travel in both directions, under the following conditions:

a) No pair of islands is joined by more than one bridge.

b) Using the bridges we can reach every island from Panacenter.

c) If we want to travel from Panacenter to every other island, in such a way that we use each bridge at most once, the number of inhabitants of the islands we visit is strictly decreasing.

Determine the number of ways we can build the bridges.

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6 Let $ABCD$ be a convex quadrilateral. $I = AC \cap BD$, and E, H, F and G are points on AB, BC, CD and DA respectively, such that $EF \cap GH = I$. If $M = EG \cap AC$, $N = HF \cap AC$, show that

$$\frac{AM}{IM} \cdot \frac{IN}{CN} = \frac{IA}{IC}.$$
