CentroAmerican 2010

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1 Denote by $S(n)$ the sum of the digits of the positive integer n. Find all the solutions of the equation

$n(S(n)-1)=2010.$

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2 Let ABC be a triangle and L, M, N be the midpoints of BC, CA and AB, respectively. The tangent to the circumcircle of ABC at A intersects LM and LN at P and Q, respectively. Show that CP is parallel to BQ.

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3 A token is placed in one square of a $m\times n$ board, and is moved according to the following rules:

-In each turn, the token can be moved to a square sharing a side with the one currently occupied.

-The token cannot be placed in a square that has already been occupied.

-Any two consecutive moves cannot have the same direction.

The game ends when the token cannot be moved. Determine the values of m and n for which, by placing the token in some square, all the squares of the board will have been occupied in the end of the game.

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4 Find all positive integers N such that an $N\times N$ board can be tiled using tiles of size 5×5 or 1×3.

Note: The tiles must completely cover all the board, with no overlappings.

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5 If p, q and r are nonzero rational numbers such that $\sqrt[3]{pq^2}+\sqrt[3]{qr^2}+\sqrt[3]{rp^2}$ is a nonzero rational number, prove that

$\frac{1}{\sqrt[3]{pq^2}}+\frac{1}{\sqrt[3]{qr^2}}+\frac{1}{\sqrt[3]{rp^2}}$

is also a rational number.

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6 Let Γ and Γ_1 be two circles internally tangent at A, with centers O and O_1 and radii r and r_1, respectively ($r>r_1$). B is a point diametrically opposed to A in Γ, and C is a point on Γ such that BC is tangent to Γ_1 at P. Let A' the midpoint of BC. Given that O_1A' is parallel to AP, find the ratio r/r_1.
