

CentroAmerican 2011
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Day 1

1 Consider a cube with a fly standing at each of its vertices. When a whistle blows, each fly moves to a vertex in the same face as the previous one but diagonally opposite to it. After the whistle blows, in how many ways can the flies change position so that there is no vertex with 2 or more flies?

2 In a scalene triangle ABC , D is the foot of the altitude through A , E is the intersection of AC with the bisector of $\angle ABC$ and F is a point on AB . Let O the circumcenter of ABC and $X = AD \cap BE$, $Y = BE \cap CF$, $Z = CF \cap AD$. If XYZ is an equilateral triangle, prove that one of the triangles OXY , OYZ , OZX must be equilateral.

3 A slip on an integer $n \geq 2$ is an operation that consists in choosing a prime divisor p of n and replacing n by $\frac{n+p^2}{p}$.

Starting with an arbitrary integer $n \geq 5$, we successively apply the slip operation on it. Show that one eventually reaches 5, no matter the slips applied.

Day 2

4 Find all positive integers p, q, r such that p and q are prime numbers and $\frac{1}{p+1} + \frac{1}{q+1} - \frac{1}{(p+1)(q+1)} = \frac{1}{r}$.

5 If x, y, z are positive numbers satisfying

$$x + \frac{y}{z} = y + \frac{z}{x} = z + \frac{x}{y} = 2.$$

Find all the possible values of $x + y + z$.

6 Let ABC be an acute triangle and D, E, F be the feet of the altitudes through A, B, C respectively. Call Y and Z the feet of the perpendicular lines from B and C to FD and DE , respectively. Let F_1 be the symmetric of F with respect to E and E_1 be the symmetric of E with respect to F . If $3EF = FD + DE$, prove that $\angle BZF_1 = \angle CYE_1$.