CentroAmerican 2012

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by hatchguy

Day 1

1 Find all positive integers that are equal to 700 times the sum of its digits.

2 Let γ be the circumcircle of the acute triangle ABC. Let P be the midpoint of the minor arc BC. The parallel to AB through P cuts BC, AC and γ at points R, S and T, respectively. Let $K \equiv AP \cap BT$ and $L \equiv BS \cap AR$. Show that KL passes through the midpoint of AB if and only if $CS = PR$.

3 Let a, b, c be real numbers that satisfy $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{a+c} = 1$ and $ab + bc + ac > 0$.

Show that

$$a + b + c - \frac{abc}{ab + bc + ac} \geq 4$$

Day 2

1 Trilandia is a very unusual city. The city has the shape of an equilateral triangle of side lenght 2012. The streets divide the city into several blocks that are shaped like equilateral triangles of side lenght 1. There are streets at the border of Trilandia too. There are 6036 streets in total. The mayor wants to put sentinel sites at some intersections of the city to monitor the streets. A sentinel site can monitor every street on which it is located. What is the smallest number of sentinel sites that are required to monitor every street of Trilandia?

2 Alexander and Louise are a pair of burglars. Every morning, Louise steals one third of Alexander's money, but feels remorse later in the afternoon and gives him half of all the money she has. If Louise has no money at the beginning and starts stealing on the first day, what is the least positive integer amount of money Alexander must have so that at the end of the 2012th day they both have an integer amount of money?

3 Let ABC be a triangle with $AB < BC$, and let E and F be points in AC and AB such that $BF = BC = CE$, both on the same halfplane as A with respect to BC.

Let G be the intersection of BE and CF. Let H be a point in the parallel through G to AC such that $HG = AF$ (with H and C in opposite halfplanes with respect to BG). Show that $\angle EHG = \frac{\angle BAC}{2}$.

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