CentroAmerican 2013

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Day 1

1 Juan writes the list of pairs $(n, 3^n)$, with $n = 1, 2, 3, ...$ on a chalkboard. As he writes the list, he underlines the pairs $(n, 3^n)$ when n and 3^n have the same units digit. What is the 2013^{th} underlined pair?

2 Around a round table the people $P_1, P_2, ..., P_{2013}$ are seated in a clockwise order. Each person starts with a certain amount of coins (possibly none); there are a total of 10000 coins. Starting with P_1 and proceeding in clockwise order, each person does the following on their turn:

-If they have an even number of coins, they give all of their coins to their neighbor to the left.

-If they have an odd number of coins, they give their neighbor to the left an odd number of coins (at least 1 and at most all of their coins) and keep the rest.

Prove that, repeating this procedure, there will necessarily be a point where one person has all of the coins.

3 Let $ABCD$ be a convex quadrilateral and let M be the midpoint of side AB. The circle passing through D and tangent to AB at A intersects the segment DM at E. The circle passing through C and tangent to AB at B intersects the segment CM at F. Suppose that the lines AF and BE intersect at a point which belongs to the perpendicular bisector of side AB. Prove that A, E, and C are collinear if and only if B, F, and D are collinear.

Day 2

1 Ana and Beatriz take turns in a game that starts with a square of side 1 drawn on an infinite grid. Each turn consists of drawing a square that does not overlap with the rectangle already drawn, in such a way that one of its sides is a (complete) side of the figure already drawn. A player wins if she completes a rectangle whose area is a multiple of 5. If Ana goes first, does either player have a winning strategy?

2 Let ABC be an acute triangle and let Γ be its circumcircle. The bisector of $\angle A$ intersects BC at D, Γ at K (different from A), and the line through B tangent to Γ at X. Show that K is the midpoint of AX if and only if $\frac{AD}{DC} = \sqrt{2}$.

3 Determine all pairs of non-constant polynomials $p(x)$ and $q(x)$, each with leading coefficient 1, degree n, and n roots which are non-negative integers, that satisfy $p(x) - q(x) = 1$.