Bulgaria National Olympiad 1998

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Day 1

1 Let n be a natural number. Find the least natural number k for which there exist k sequences of 0 and 1 of length $2n+2$ with the following property: any sequence of 0 and 1 of length $2n+2$ coincides with some of these k sequences in at least $n+2$ positions.

2 The polynomials $P_n(x,y), n=1,2,...$ are defined by

$$P_1(x,y)=1, P_{n+1}(x,y)=(x+y-1)(y+1)P_n(x,y+2)+(y-y^2)P_n(x,y)$$

Prove that $P_n(x,y)=P_n(y,x)$ for all $x,y\in\mathbb{R}$ and n.

3 On the sides of a non-obtuse triangle ABC a square, a regular n-gon and a regular m-gon (m,$n>5$) are constructed externally, so that their centers are vertices of a regular triangle. Prove that $m=n=6$ and find the angles of $\triangle ABC$.

Day 2

1 Let $a_1,a_2,\cdots,a_n$ be real numbers, not all zero. Prove that the equation:

$$\sqrt{1+a_1x}+\sqrt{1+a_2x}+\cdots+\sqrt{1+a_nx}=n$$

has at most one real nonzero root.

2 let m and n be natural numbers such that: $3m|(m+3)^n+1$
Prove that $\frac{(m+3)^n+1}{3m}$ is odd

3 The sides and diagonals of a regular n-gon R are colored in k colors so that:
(i) For each color a and any two vertices A,B of R , the segment AB is of color a or there is a vertex C such that AC and BC are of color a.
(ii) The sides of any triangle with vertices at vertices of R are colored in at most two colors.
Prove that $k\le 2$.