Bulgaria National Olympiad 1999

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Day 1

1	The faces of a box with integer edge lengths are painted green. The box is partitioned into unit cubes. Find the dimensions of the box if the number of unit cubes with no green face is one third of the total number of cubes.
2	Let $\{a_n\}$ be a sequence of integers satisfying $(n-1)a_{n+1}=(n+1)a_n-2(n-1)\forall n\ge 1$. If $2000\vert a_{1999}$, find the smallest $n\ge 2$ such that $2000\vert a_n$.
3	The vertices of a triangle have integer coordinates and one of its sides is of length $\sqrt{n}$, where n is a square-free natural number. Prove that the ratio of the circumradius and the inradius is an irrational number.

Day 2

1	Find the number of all integers n with $4\le n\le 1023$ which contain no three consecutive equal digits in their binary representations.
2	The vertices A,B,C of an acute-angled triangle ABC lie on the sides B1C1, C1A1, A1B1 respectively of a triangle A1B1C1 similar to the triangle ABC (A = A1, etc.). Prove that the orthocenters of triangles ABC and A1B1C1 are equidistant from the circumcenter of ABC.
3	Prove that $x^3+y^3+z^3+t^3=1999$ has infinitely many soln. over $\mathbb{Z}$.