Bulgaria National Olympiad 2000

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Day 1

1 In the coordinate plane, a set of 2000 points $\{(x_1,y_1),(x_2,y_2),...,(x_{2000},y_{2000})\}$ is called *good* if $0\leq x_i\leq 83, 0\leq y_i\leq 83$ for $i=1,2,\dots,2000$ and $x_i\neq x_j$ when $i\neq j$. Find the largest positive integer n such that, for any good set, the interior and boundary of some unit square contains exactly n of the points in the set on its interior or its boundary.

2 Let be given an acute triangle ABC. Show that there exist unique points $A_1\in BC$, $B_1\in CA$, $C_1\in AB$ such that each of these three points is the midpoint of the segment whose endpoints are the orthogonal projections of the other two points on the corresponding side. Prove that the triangle $A_1B_1C_1$ is similar to the triangle whose side lengths are the medians of $\triangle ABC$.

3 Let p be a prime number and let $a_1,a_2,\dots,a_{p-2}$ be positive integers such that p doesn't a_k or ${a_k}^k-1$ for any k. Prove that the product of some of the a_i's is congruent to 2 modulo p.

Day 2

1 Find all polynomials $P(x)$ with real coefficients such that

$$P(x)P(x+1)=P(x^2),\quad \forall x\in\mathbb{R}.$$

2 Let D be the midpoint of the base AB of the isosceles acute triangle ABC. Choose point E on segment AB, and let O be the circumcenter of triangle ACE. Prove that the line through D perpendicular to DO, the line through E perpendicular to BC, and the line through B parallel to AC are concurrent.

3 Let A be the set of all binary sequences of length n and denote $o=(0,0,\dots,0)\in A$. Define the addition on A as $(a_1,\dots,a_n)+(b_1,\dots,b_n)=(c_1,\dots,c_n)$, where $c_i=0$ when $a_i=b_i$ and $c_i=1$ otherwise. Suppose that $f\colon A\to A$ is a function such that $f(0)=0$, and for each $a,b\in A$, the sequences $f(a)$ and $f(b)$ differ in exactly as many places as a and b do. Prove that if a , b, $c\in A$ satisfy $a+b+c=0$, then $f(a)+f(b)+f(c)=0$.