Bulgaria National Olympiad 2001

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Day 1

1 Consider the sequence $\{a_n\}$ such that $a_0=4$, $a_1=22$, and $a_n-6a_{n-1}+a_{n-2}=0$ for $n\geq 2$. Prove that there exist sequences $\{x_n\}$ and $\{y_n\}$ of positive integers such that

$$a_n=\frac{y_n^2+7}{x_n-y_n}$$

for any $n\geq 0$.

2 Suppose that $ABCD$ is a parallelogram such that $DAB>90$. Let the point H to be on AD such that BH is perpendicular to AD. Let the point M to be the midpoint of AB. Let the point K to be the intersecting point of the line DM with the circumcircle of ADB. Prove that $HKCD$ is concyclic.

3 Given a permutation $(a_1, a_1, ..., a_n)$ of the numbers $1, 2, ..., n$ one may interchange any two consecutive "blocks" - that is, one may transform

$(a_1, a_2, ..., a_i, \underbrace{a_{i+1}, ... a_{i+p},}_{A} \underbrace{a_{i+p+1}, ..., a_{i+q},}_{B} ..., a_n)$

into $(a_1, a_2, ..., a_i, \underbrace{a_{i+p+1}, ..., a_{i+q},}_{B} \underbrace{a_{i+1}, ... a_{i+p},}_{A} ..., a_n)$

by interchanging the "blocks" A and B. Find the least number of such changes which are needed to transform $(n, n-1, ..., 1)$ into $(1, 2, ..., n)$

Day 2

1 Let $n\geq 2$ be a given integer. At any point (i,j) with $i,j\in\mathbb{Z}$ we write the remainder of $i+j$ modulo n. Find all pairs (a,b) of positive integers such that the rectangle with vertices $(0,0)$, $(a,0)$, (a,b), $(0,b)$ has the following properties:
(i) the remainders $0,1,\dots,n-1$ written at its interior points appear the same number of times;
(ii) the remainders $0,1,\dots,n-1$ written at its boundary points appear the same number of times.

2 Find all real values t for which there exist real numbers x, y, z satisfying : $3x^2+3xz+z^2=1$,
$3y^2+3yz+z^2=4$,
$x^2-xy+y^2=t$.

3 Let p be a prime number congruent to 3 modulo 4, and consider the equation $(p+2)x^2-(p+1)y^2+px+(p+2)y=1$.
Prove that this equation has infinitely many solutions in positive integers, and show that if $(x,y)=(x_0,y_0)$ is a solution of the equation in positive integers, then $p|x_0$.
