Bulgaria National Olympiad 2003

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Day 1 May 17th

1 Let $x_1, x_2 \ldots, x_5$ be real numbers. Find the least positive integer n with the following property: if some n distinct sums of the form $x_p + x_q + x_r$ (with $1 \le p < q < r \le 5$) are equal to 0, then $x_1 = x_2 = \cdots = x_5 = 0$.

2 Let H be an arbitrary point on the altitude CP of the acute triangle ABC. The lines AH and BH intersect BC and AC in M and N, respectively.

(a) Prove that $\angle NPC = \angle MPC$.
(b) Let O be the common point of MN and CP. An arbitrary line through O meets the sides of quadrilateral $CNHM$ in D and E. Prove that $\angle EPC = \angle DPC$.

3 Given the sequence $\{y_n\}_{n=1}^{\infty}$ defined by $y_1 = y_2 = 1$ and

$$y_{n+2} = (4k-5)y_{n+1} - y_n + 4 - 2k, \qquad n \ge 1$$

find all integers k such that every term of the sequence is a perfect square.

Day 2 May 18th

1 A set A of positive integers is called *uniform* if, after any of its elements removed, the remaining ones can be partitioned into two subsets with equal sum of their elements. Find the least positive integer $n > 1$ such that there exist a uniform set A with n elements.

2 Let a, b, c be rational numbers such that $a + b + c$ and $a^2 + b^2 + c^2$ are **equal** integers. Prove that the number abc can be written as the ratio of a perfect cube and a perfect square which are relatively prime.

3 Determine all polynomials $P(x)$ with integer coefficients such that, for any positive integer n, the equation $P(x) = 2^n$ has an integer root.