Bulgaria National Olympiad 2004

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Day 1

1 Let I be the incenter of triangle ABC, and let A_1, B_1, C_1 be arbitrary points on the segments (AI), (BI), (CI), respectively. The perpendicular bisectors of AA_1, BB_1, CC_1 intersect each other at A_2, B_2, and C_2. Prove that the circumcenter of the triangle $A_2B_2C_2$ coincides with the circumcenter of the triangle ABC if and only if I is the orthocenter of triangle $A_1B_1C_1$.

2 For any positive integer n the sum $1+\frac{1}{2}+\cdots+\frac{1}{n}$ is written in the form $\frac{P(n)}{Q(n)}$, where $P(n)$ and $Q(n)$ are relatively prime.

a) Prove that $P(67)$ is not divisible by 3;

b) Find all possible n, for which $P(n)$ is divisible by 3.

3 A group consist of n tourists. Among every 3 of them there are 2 which are not familiar. For every partition of the tourists in 2 buses you can find 2 tourists that are in the same bus and are familiar with each other. Prove that is a tourist familiar to at most $\frac{2}{5}n$ tourists.

Day 2

4 In a word formed with the letters a, b we can change some blocks: aba in b and back, bba in a and backwards. If the initial word is $aaa\ldots ab$ where a appears 2003 times can we reach the word $baaa\ldots a$, where a appears 2003 times.

5 Let a, b, c, d be positive integers such that the number of pairs $(x, y) \in (0,1)^2$ such that both $ax+by$ and $cx+dy$ are integers is equal with 2004. If $\gcd(a, c) = 6$ find $\gcd(b, d)$.

6 Let p be a prime number and let $0 \le a_1 < a_2 < \cdots < a_m < p$ and $0 \le b_1 < b_2 < \cdots < b_n < p$ be arbitrary integers. Let k be the number of distinct residues modulo p that $a_i + b_j$ give when i runs from 1 to m, and j from 1 to n. Prove that

a) if $m+n > p$ then $k = p$;

b) if $m+n \le p$ then $k \ge m+n-1$.