Bulgaria National Olympiad 2005

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Day 1 May 14th

1 Determine all triples (x,y,z) of positive integers for which the number

$$\sqrt{\frac{2005}{x+y}}+\sqrt{\frac{2005}{y+z}}+\sqrt{\frac{2005}{z+x}}$$

is an integer .

2 Consider two circles k_1, k_2 touching externally at point T. a line touches k_2 at point X and intersects k_1 at points A and B. Let S be the second intersection point of k_1 with the line XT . On the arc $\widehat{TS}$ not containing A and B is chosen a point C . Let CY be the tangent line to k_2 with $Y \in k_2$, such that the segment CY does not intersect the segment ST . If $I = XY \cap SC$. Prove that :

(a) the points C, T, Y, I are concyclic.

(b) I is the excenter of triangle ABC with respect to the side BC.

3 Let $M = (0,1) \cap \mathbb{Q}$. Determine, with proof, whether there exists a subset $A \subset M$ with the property that every number in M can be uniquely written as the sum of finitely many distinct elements of A.

Day 2 May 15th

4 Let ABC be a triangle with $AC \neq BC$, and let $A'B'C$ be a triangle obtained from ABC after some rotation centered at C. Let M, E, F be the midpoints of the segments BA', AC and CB' respectively. If $EM = FM$, find $\widehat{EMF}$.

5 For positive integers t,a,b,a (t,a,b)-*game* is a two player game defined by the following rules. Initially, the number t is written on a blackboard. At his first move, the 1st player replaces t with either $t-a$ or $t-b$. Then, the 2nd player subtracts either a or b from this number, and writes the result on the blackboard, erasing the old number. After this, the first player once again erases either a or b from the number written on the blackboard, and so on. The player who first reaches a negative number loses the game. Prove that there exist infinitely many values of t for which the first player has a winning strategy for all pairs (a,b) with $a+b=2005$.

6 Let a, b and c be positive integers such that ab divides $c(c^2 - c + 1)$ and $a + b$ is divisible by $c^2 + 1$.
Prove that the sets $\{a, b\}$ and $\{c, c^2 - c + 1\}$ coincide.