**Bulgaria National Olympiad 2006**

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by bilarev, imortal

**Day 1**

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**1** Consider the set $A=\{1,2,3\ldots,2^n\}, n\geq 2$. Find the number of subsets $B$ of $A$ such that for any two elements of $A$ whose sum is a power of $2$ exactly one of them is in $B$.

*Aleksandar Ivanov*

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**2** Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be a function that satisfies for all $x>y>0$

$$f(x+y)-f(x-y)=4\sqrt{f(x)f(y)}$$

a) Prove that $f(2x)=4f(x)$ for all $x>0$;
b) Find all such functions.

*Nikolai Nikolov, Oleg Mushkarov*

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**3** The natural numbers are written in sequence, in increasing order, and by this we get an infinite sequence of digits. Find the least natural $k$, for which among the first $k$ digits of this sequence, any two nonzero digits have been written a different number of times.

*Aleksandar Ivanov, Emil Kolev*

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**Day 2**

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**1** Let $p$ be a prime such that $p^2$ divides $2^{p-1}-1$. Prove that for all positive integers $n$ the number $\left(p-1\right)\left(p!+2^n\right)$ has at least $3$ different prime divisors.

*Aleksandar Ivanov*

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**2** The triangle $ABC$ is such that $\angle BAC=30^{\circ}, \angle ABC=45^{\circ}$. Prove that if $X$ lies on the ray $AC$, $Y$ lies on the ray $BC$ and $OX=BY$, where $O$ is the circumcentre of triangle $ABC$, then $S_{XY}$ passes through a fixed point.

*Emil Kolev*

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**3** Consider a point $O$ in the plane. Find all sets $S$ of at least two points in the plane such that if $A\in S$ ad $A\neq O$, then the circle with diameter $OA$ is in $S$.

*Nikolai Nikolov, Slavomir Dinev*

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