Bulgaria National Olympiad 2007

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Day 1

1 The quadrilateral $ABCD$, where $\angle BAD + \angle ADC > \pi$, is inscribed a circle with centre I. A line through I intersects AB and CD in points X and Y respectively such that $IX = IY$. Prove that $AX \cdot DY = BX \cdot CY$.

2 Find the greatest positive integer n such that we can choose 2007 different positive integers from $[2 \cdot 10^{n-1}, 10^{n})$ such that for each two $1 \leq i < j \leq n$ there exists a positive integer $\overline{a_1 a_2 \ldots a_n}$ from the chosen integers for which $a_j \geq a_i + 2$.

A. Ivanov, E. Kolev

3 Find the least positive integer n such that $\cos \frac{\pi}{n}$ cannot be written in the form $p + \sqrt{q} + \sqrt[3]{r}$ with $p, q, r \in \mathbb{Q}$.

O. Mushkarov, N. Nikolov

No-one in the competition scored more than 2 points

Day 2

1 Let $k > 1$ be a given positive integer. A set S of positive integers is called *good* if we can colour the set of positive integers in k colours such that each integer of S cannot be represented as sum of two positive integers of the same colour. Find the greatest t such that the set $S = \{a+1, a+2, \ldots, a+t\}$ is *good* for all positive integers a.

A. Ivanov, E. Kolev

2 Find the least real number m such that with all 5 equilateral triangles with sum of areas m we can cover an equilateral triangle with side 1.

O. Mushkarov, N. Nikolov

3 Let $P(x) \in \mathbb{Z}[x]$ be a monic polynomial with even degree. Prove that, if for infinitely many integers x, the number $P(x)$ is a square of a positive integer, then there exists a polynomial $Q(x) \in \mathbb{Z}[x]$ such that $P(x) = Q(x)^2$.